Linear Algebra
[KOMS119602] - 2022/2023

10.1 - Relation between Vectors in a Space

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Learning objectives

After this lecture, you should be able to:

1. explain the concept of spanning set and linear combination of vectors;
2. explain the concept of basis and dimension of vector space;
3. find a basis and the dimension of a vector space.

Subspace and Linear Combination

Linear combination

Recall that linear combination of vectors is defined as:

> *Let* $\mathbf{w} \in V$*. Then* w *is a linear combination of vectors* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ *if* $\mathbf{w}$ *can be written as:*
>
> $$\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n$$
>
> *where* $k_1, k_2, \ldots, kn \in \mathbb{R}$*.*

Example

Let $\mathbf{v}_1 = (3, 2, -1)$ and $\mathbf{v}_2 = (2, -4, 3)$. Then:

$$\mathbf{w} = 2\mathbf{v}_1 + 3\mathbf{v}_2 = 2(3, 2, -1) + 3(2, -4, 3) = (12, -8, 7)$$

is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

Defining linear combination of vectors

Given a vector $(5, 9, 5)$. How to represent the vector as a linear combination of vectors:

$$\mathbf{u} = (2, 1, 4),\ \mathbf{v} = (1, -1, 3),\ \text{and } \mathbf{w} = (3, 2, 5)$$

Solution: Let $k_1, k_2, k_3 \in \mathbb{R}$ be such that:

$$k_1 \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} + k_3 \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \\ 5 \end{bmatrix}$$

This yields linear system:

$$\begin{cases} 2k_1 + \ \ k_2 + 3k_3 = 3 \\ \ \ k_1 - \ \ k_2 + 2k_3 = 9 \\ 4k_1 + 3k_2 + 5k_3 = 5 \end{cases}$$

By Gauss elimination, we obtain:

$$k_1 = 3,\ k_2 = -4,\ k_3 = 2$$

Linear combination forms subspace

Theorem

If $S = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\}$ is a set of vectors in a vector space V. Then:

1. *The set W containing all linear combinations of vectors in S is a subspace of V.*
2. *W is the smallest subspace of V that contains vectors in S, i.e., all the other subspaces containing the vectors also contain W.*

Exercise: prove the correctness of the theorem.

Spanning Set

Set of vectors forming subspace

- Let V be a vector space, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in V$.
- Let W be a subspace of V s.t. $\forall \mathbf{w} \in W$,

$$\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n$$

where $k_1, k_2, \ldots, k_n$ are scalars.

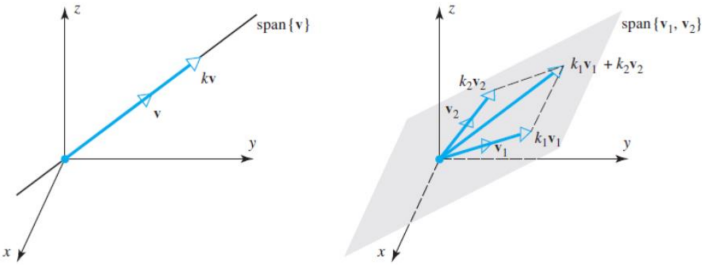
Hence, $= \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is said to span W.
S is called spanning set, and is denoted as:

$$span\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \quad \text{or} \quad span(S)$$

Example: *space spanned by one of two vectors*

Let $\mathbf{v}_1, \mathbf{v}_2$ are *noncollinear* vectors in $\mathbb{R}^3$, with their initial points at the origin, then:

- *span*$\{\mathbf{v}_1, \mathbf{v}_2\}$ consisting all linear combinations $k_1\mathbf{v}_1 + k_2\mathbf{v}_2$, is the plane determined by vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.
- if $\mathbf{v} \neq \mathbf{0}$ is a vector in $\mathbb{R}^2$ or $\mathbb{R}^3$, then *span*$\{\mathbf{v}\}$ consisting all scalar multiples $k\mathbf{v}$, is the line determined by $\mathbf{v}$.



(*a*) Span{**v**} is the line through the origin determined by **v**.

(*b*) Span{$\mathbf{v}_1$, $\mathbf{v}_2$} is the plane through the origin determined by $\mathbf{v}_1$ and $\mathbf{v}_2$.

Exercise 1

The following standard unit vectors span $\mathbb{R}^3$.

$$\mathbf{i} = (1, 0, 0),\ \mathbf{j} = (0, 1, 0),\ \mathbf{k} = (0, 0, 1)$$

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This is because, every vector $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$ can be represented as linear combination:

$$\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$

In this case, $\mathbb{R}^3 = span\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$.

Exercise 2

Polynomials $1, x, x^2, \ldots, x^n$ *span the vector space* P_n

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Polynomials $1, x, x^2, \ldots, x^n$ *span the vector space* P_n

This is because, every polynomial $\mathbf{p} \in P_n$ can be written as:

$$\mathbf{p} = a_0 + a_1 x + a_2 x_2 + \cdots + a_n x^n$$

which is a linear combination of $1, x, x^2, \ldots, x^n$.

In this case, $P_n = span\{1, x, x^2, \ldots, x^n\}$.

Exercise 3

Determine whether following vectors span $\mathbb{R}^3$ *!*

$$\mathbf{v}_1 = (2, -1, 3),\ \mathbf{v}_2 = (4, 1, 2),\ \mathbf{v}_3 = (8, -1, 8)$$

Exercise 3

Determine whether following vectors span $\mathbb{R}^3$!

$$\mathbf{v}_1 = (2, -1, 3),\ \ \mathbf{v}_2 = (4, 1, 2),\ \ \mathbf{v}_3 = (8, -1, 8)$$

Let $\mathbf{u} = (u_1, u_2, u_3)$ be a vector in $\mathbb{R}^3$, and $k_1, k_2, k_3 \in \mathbb{R}$.

If the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ span $\mathbb{R}^3$, then it should be:

$$(u_1, u_2, u_3) = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3$$

We will check if the following linear system has a solution.

$$\begin{cases} 2k_1 + 4k_2 + 8k_3 & = u_1 \\ -k_1 + k_2 - k_3 & = u_2 \\ 3k_1 + 2k_2 + 8k_3 & = u_3 \end{cases}$$

Exercise 4 (*cont.*)

The linear system has coefficient matrix:

$$A = \begin{bmatrix} 2 & 4 & 8 \\ -1 & 1 & -1 \\ 3 & 2 & 8 \end{bmatrix}$$

Note that:

$$\det(A) = 2\begin{vmatrix} 1 & -1 \\ 2 & 8 \end{vmatrix} - 4\begin{vmatrix} -1 & -1 \\ 3 & 8 \end{vmatrix} + 8\begin{vmatrix} -1 & 1 \\ 3 & 2 \end{vmatrix} = 20 + 20 - 40 = 0$$

Hence, there is no solution for the linear system, meaning that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ does not span $\mathbb{R}^3$.

Linear Independence

Linear independence in $\mathbb{R}^2$ and $\mathbb{R}^3$

Let V be a vector space. The set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is said linearly independent iff the linear equation

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n = 0 \tag{1}$$

has exactly one solution, which is the trivial solution:

$$k_1 = 0,\ k_2 = 0,\ \ldots, k_n = 0$$

Conversely, the set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is said not linearly independent or linearly dependent, iff the linear combination (1) has a non-trivial solution (i.e., a solution other than $k_1 = 0,\ k_2 = 0,\ \ldots, k_n = 0$).

Example of linearly independent set

The vectors $\mathbf{i} = (1, 0, 0)$*,* $\mathbf{j} = (0, 1, 0)$*, and* $\mathbf{k} = (0, 0, 1)$ *are linearly independent vectors in* $\mathbb{R}^3$*.*

Why?

Note that for scalars $k_1, k_2, k_3 \in \mathbb{R}$, we have: $k_1\mathbf{i} + k_2\mathbf{j} + k_3\mathbf{k} = \mathbf{0}$,

that is equivalent to

$$k_1(1, 0, 0) + k_2(0, 1, 0) + k_3(0, 0, 1) = (0, 0, 0) \Leftrightarrow (k_1, k_2, k_3) = (0, 0, 0)$$

Clearly, there is no solution other than $k_1 = 0$, $k_2 = 0$, and $k_3 = 0$.

This means that $S = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is linearly independent.

Similarly, we can show that:

$$\mathbf{e}_1 = (1, 0, 0, \ldots, 0), \ \mathbf{e}_2 = (0, 1, 0, \ldots, 0), \text{ and } \mathbf{e}_n = (0, 0, 0, \ldots, 1)$$

are linearly independent vectors.

Example of linearly dependent sets (1)

Determine whether the vectors:

$$\mathbf{v}_1 = (2, -1, 0, 3),\ \mathbf{v}_2 = (1, 2, 5, -1),\ \text{ and }\ \mathbf{v}_3 = (7, -1, 5, 8)$$

are linearly independent or not!

Example of linearly dependent sets (1)

Determine whether the vectors:

$$\mathbf{v}_1 = (2, -1, 0, 3),\ \mathbf{v}_2 = (1, 2, 5, -1), \quad \text{and} \quad \mathbf{v}_3 = (7, -1, 5, 8)$$

are linearly independent or not!

Solution:

Note that: $3\mathbf{v}_1 + v_2 - v_3 = \mathbf{0}$ (*show it!*).

This means that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is **not** linearly independent.

Example of linearly dependent sets (2)

Determine if the polynomials:

$$\mathbf{p}_1 = 1 - x, \quad \mathbf{p}_2 = 5 + 3x - 2x^2, \quad \text{and} \quad \mathbf{p}_3 = 1 + 3x - x^2$$

are linearly independent or not!

Example of linearly dependent sets (2)

Determine if the polynomials:

$$\mathbf{p}_1 = 1 - x, \quad \mathbf{p}_2 = 5 + 3x - 2x^2, \quad \text{and} \quad \mathbf{p}_3 = 1 + 3x - x^2$$

are linearly independent or not!

Solution:

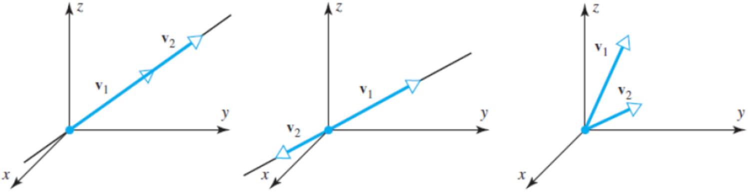
Note that $3\mathbf{p}_1 - \mathbf{p}_2 + 2\mathbf{p}_3 = \mathbf{0}$ (*show it!*).

Hence, the vectors are linearly dependent.

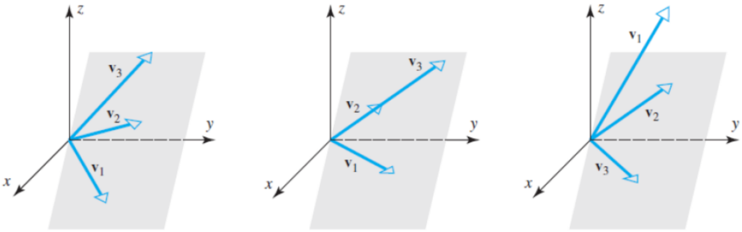
Exercises

Do the relevant exercises in the Howard Anton's nook.

Geometric interpretation of linear independence in $\mathbb{R}^2$ and $\mathbb{R}^3$



(a) Linearly dependent (b) Linearly dependent (c) Linearly independent



(a) Linearly dependent (b) Linearly dependent (c) Linearly independent

Determining linear independence/dependence (1)

Determine the linear dependence of the vectors:

$$\mathbf{v}_1 = (1, -2, 3),\ \mathbf{v}_2 = (5, 6, -1),\ \text{ and }\ \mathbf{v}_3 = (3, 2, 1)$$

Solution:

We check if the vector equation $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0}$ has a solution in $\mathbb{R}$.
The equation is equivalent to:

$$\begin{aligned} k_1(1, -2, 3) + k_2(5, 6, -1) + k_3(3, 2, 1) &= (0, 0, 0) \\ (k_1 + 5k_2 + 3k_3, -2k_1 + 6k_2 + 2k_3, 3k_1 - k_2 + k_3) &= (0, 0, 0) \end{aligned}$$

Solve the system:

$$\begin{cases} k_1 + 5k_2 + 3k_3 = 0 \\ 2k_1 + 6k_2 + 2k_3 = 0 \\ 3k_1 - k_2 + k_3 = 0 \end{cases}$$

Solving the system using Gaussian elimination, we get:

$$k_1 = -\frac{1}{2}t,\ k_2 = -\frac{1}{2}t,\ k_3 = t, \qquad t \in \mathbb{R}$$

Hence, the system has a non-trivial solution, so the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent.

Determining linear independence/dependence (2)

Show that the polynomials form a linearly independent set of vectors in P_n.

$$1, x, x^2, \ldots, x^n$$

Determining linear independence/dependence (2)

Show that the polynomials form a linearly independent set of vectors in P_n.

$$1, x, x^2, \ldots, x^n$$

Solution:

Let $a_0, a_1, \ldots, a_n$ be such that:

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = \mathbf{0}$$

We must show that the only solution of the polynomial for $x \in (-\infty, \infty)$ is:

$$a_0 = a_1 = a_2 = \cdots = a_n = 0$$

From Algebra, we know that:

Theorem

Every nonzero polynomial of degree n has at most n roots.

This implies that $a_0 = a_1 = \cdots = a_n$ (or, the polynomial is zero polynomial).

Otherwise, it is a nonzero polynomial, having infinite number of roots (that is, $x \in (-\infty, \infty)$), contradicting the theorem.

Exercises

Do the relevant exercises in Howard Antons' book.